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Monomial Golod Quotients of Exterior Algebras

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It is proved that for any set M of squarefree monomials in the variables $x_1, \ldots, x_n$, the algebra $A = k[x_1, \ldots, x_n]/(M)$ is Golod if and only if the algebra $B = E(x_1, \ldots, x_n)/(M)$ is Golod, where E is the exterior algebra. This is proved by showing the equivalence of the extremality of the Poincaré series of A and B.

1. INTRODUCTION

Let Δ be a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$. To each such Δ one can associate an ideal I_Δ in the polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ by letting I_Δ be generated by the set of monomials

$$\left\{ x_{i_1} x_{i_2} \cdots x_{i_m} \,|\, \{i_1, i_2, \ldots, i_m\} \notin \Delta \right\}. \tag{1}$$

Conversely, every ideal in S generated by squarefree monomials comes from some simplicial complex in this way. The quotient ring S/I_Δ is then called the Stanley–Reisner ring of Δ, and is denoted by $k[\Delta]$. If we denote the exterior algebra over k on the variables $x_1, x_2, \ldots, x_n$ by E, we may form the ideal J_Δ in E by letting it be generated by the set (1). The quotient E/J_Δ is now denoted by $k\{\Delta\}$, and is called the indicator algebra of Δ.

The relations between $k[\Delta]$ and $k\{\Delta\}$, or between more general quotients S/I and E/J, have been investigated from several viewpoints, some of which we will recall:

- Macaulay's theorem, which gives an upper bound for the possible Hilbert series of S/I, has a counterpart in the Kruskal–Katona theorem which gives a similar bound for the Hilbert series of E/J.

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• Gotzmann's theorem, which essentially says that an equality in Macaulay's bound in a degree will persist in higher degrees, if no new relations are introduced, has a counterpart for quotients of the exterior algebra as well. This is proved by Aramova *et al.* [AHH97].

• The correspondence between S-resolutions of $k[\Delta]$ and E-resolutions of $k\{\Delta\}$ has been studied by among others Aramova *et al.* [AAH]. In particular they give the relation between $\dim_k \operatorname{Tor}_*^S(k[\Delta], k)$ and $\dim_k \operatorname{Tor}_*^E(k\{\Delta\}, k)$.

The purpose of the present paper is to relate the properties of $\operatorname{Ext}^*_{k[\Delta]}(k, k)$ and $\operatorname{Ext}^*_{k\{\Delta\}}(k, k)$ to each other in one aspect, namely to prove that $k[\Delta]$ is Golod if, and only if, $k\{\Delta\}$ is Golod. Since there are classes of Stanley–Reisner rings which are known to be Golod, (e.g., when I_Δ is a squarefree lexsegment ideal [AHH98], or more generally, when I_Δ is squarefree 0-Borel fixed [Pee96], this gives new examples of Golod quotients of exterior algebras.

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2. NOTATION AND DEFINITIONS

We will let k be an arbitrary field, and we will consider finitely generated $\mathbb{N} \times \mathbb{Z}_2$-graded k-algebras. All algebras will also be supercommutative, i.e., for homogeneous elements a and b,

$$ab = (-1)^{|a||b|} ba,$$

where $|\cdot|$ is the $\mathbb{Z}_2$-degree. Our algebras will furthermore be connected. Prime examples from this class of algebras are the polynomial ring $k[x_1, \ldots, x_n]$, where $\deg x_i = (1, 0)$, and the exterior algebra $E(x_1, \ldots, x_n)$, where $\deg x_i = (1, 1)$, and graded quotients of these.

The reason for considering this class is that Tate and Gulliksen's [Tat57, Gul68] construction of minimal resolutions of k which have a DGA structure applies. If W is a DGA, we will use the standard notation

$$\tilde{Z}(W) = \operatorname{Ker}(Z(W) \to k),$$

$$\tilde{H}(W) = \operatorname{Ker}(H(W) \to k),$$

$$IW = \operatorname{Ker}(W \to k).$$

For an $\mathbb{N}^n$-graded algebra A, and $\mathbb{N}^n$-graded A-module M, the Poincaré series of M is a formal power series defined by

$$\mathrm{P}_A^M(y_1,\ldots,y_n,z) = \mathrm{P}_A^M(\mathbf{y},z) = \sum_{i,j} \dim_k \operatorname{Ext}_A^{i,j}(M,k)\mathbf{y}^j z^i.$$

Note that the $\mathbb{Z}_2$-grading is not taken into account in this definition.

DEFINITION 1. Let R be an $\mathbb{N}\times\mathbb{Z}_2$-graded connected supercommutative k-algebra, and I be a homogeneous ideal and let $f\colon R \to R/I = \overline{R}$ be the natural projection. Then f is said to be a *Golod homomorphism* if the following condition is satisfied:

$$\mathrm{P}_{\overline{R}}^k(y,z) = \frac{\mathrm{P}_R^k(y,z)}{1 - z^2\mathrm{P}_R^I(y,z)}.$$

Remark 1. The definition is from Levin [Lev76], who assumed R to be a commutative local ring. The situation in the supercommutative situation is a special case of the definition of Golod homomorphisms of DGA:s given by Avramov [Avr86]. Levin also gives the following sufficient condition for a projection $R \to R/I$ to be a Golod homomorphism: let X be a minimal DGA-resolution of k over R, then, according to [Lev76, Lemma 1.6], the existence of an R/I-submodule V of $X \otimes_R R_+/I$ such that $V^2 = 0$ and $\tilde{Z}(X \otimes_R R/I) \subset V + B(X \otimes_R R/I)$, implies that the projection is a Golod homomorphism. This condition will be used in the proof of Lemma 3.

DEFINITION 2. Let $I \subset S$ or $J \subset E$ be a homogeneous ideal. The quotient S/I or E/J is said to be a *Golod ring* if $S \to S/I$ or $E \to E/J$ is a Golod homomorphism.

3. THE MAIN RESULT

We will now state the main result of the paper.

THEOREM 1. *Let Δ be a finite simplicial complex; then $k[\Delta]$ is a Golod ring if and only if $k\{\Delta\}$ is a Golod ring.*

The proof depends on some lemmas that we will obtain before proving the theorem. The first of these lemmas is due to Aramova, Avramov, and Herzog; we will just reformulate it to suit our needs.

LEMMA 1. *The power series* $\mathrm{P}_E^{k\{\Delta\}}$ *and* $\mathrm{P}_S^{k[\Delta]}$ *are related by*

$$\mathrm{P}_E^{k\{\Delta\}}(y_1, \ldots, y_n, z) = \mathrm{P}_S^{k[\Delta]}\left(\frac{y_1}{1 - y_1 z}, \cdots, \frac{y_n}{1 - y_n z}, z\right).$$

Proof. For $i \in \mathbb{N}$ and $\mathbf{j} \in \mathbb{N}^n$, define $\beta_{i,\mathbf{j}}$ by $\mathrm{P}_S^{k[\Delta]} = \sum_{i,\mathbf{j}} \beta_{i,\mathbf{j}} \mathbf{y}^{\mathbf{j}} z^i$. According to [AAH, Proposition 2.1],

$$\mathrm{P}_E^{k\{\Delta\}} = \sum_{i,\mathbf{j}} \beta_{i,\mathbf{j}} \frac{\mathbf{y}^{\mathbf{j}}}{\prod_{\nu, j_\nu \neq 0}(1 - y_\nu z)} z^i.$$

The homology of $k[\Delta]$ is only nonzero in squarefree multidegrees, i.e., $\beta_{i,\mathbf{j}} = 0$ if any $j_\nu \notin \{0, 1\}$ (see Fröberg [Frö82]); this implies the desired formula. ∎

The other main ingredient of the proof of Theorem 1 is to replace computations over $k\{\Delta\}$ with a series of computations over the rings $R_i = k[\Delta]/(x_1^2, \ldots, x_i^2)$; we think of $k[\overline{\Delta}] = R_n$ as being the "Artinification" of $k[\Delta]$.

Let $\mathscr{C}$ denote the category of multigraded $k[\overline{\Delta}]$-modules, i.e., the objects are finitely generated $\mathbb{N}^n$-graded modules, and the morphisms are $k[\overline{\Delta}]$-linear maps $f\colon M \to N$ such that $f(M_{\mathbf{j}}) \subset N_{\mathbf{j}}$, for $\mathbf{j} \in \mathbb{N}^n$. Similarly, let $\mathscr{D}$ denote the category of multigraded left $k\{\Delta\}$-modules.

Define a functor $F\colon \mathscr{C} \to \mathscr{D}$ in the following way. For a module M in $\mathscr{C}$ define $F(M)$ by letting it have the same underlying k-space as M, and for a homogeneous element $m \in M$ of multidegree $(\alpha_1, \ldots, \alpha_n)$ we define the $k\{\Delta\}$-action $x_i \circ m$ on m by

$$x_i \circ m = (-1)^{\sum_{j<i} \alpha_j} x_i \bullet m,$$

where $x_i \bullet m$ is the $k[\overline{\Delta}]$-action on m. It is now easy to see that this is indeed a $k\{\Delta\}$-action. For a morphism ϕ we define $F(\phi) := \phi$.

Similarly, we may define another functor $G\colon \mathscr{D} \to \mathscr{C}$, by (for N in $\mathscr{C}$) letting $G(N)$ have the same k-structure as N, and define

$$x_i \bullet n = (-1)^{\sum_{j<i} \alpha_j} x_i \circ n$$

for $n \in N$ a homogeneous element of multidegree $(a_1, \ldots, a_n)$. As before $G(\psi) = \psi$.

PROPOSITION 1. *The functors F and G defined above are additive, exact, mutually quasi-inverse equivalences of categories that commute with* Ext*s*.

Proof. It is clear that F and G are additive, exact, and quasi-inverse. If $\mathbf{P}$ is a multigraded $k[\overline{\Delta}]$-free resolution of K, then $F(\mathbf{P})$ is a multigraded $k\{\Delta\}$-free resolution of $F(K)$, and the complexes of multigraded vector spaces $\mathrm{Hom}_{k\{\Delta\}}(F(\mathbf{P}), F(L))$ and $\mathrm{Hom}_{k[\overline{\Delta}]}(\mathbf{P}, L)$ are equal. This implies an equality of Exts. ∎

As an immediate consequence of the proposition, we obtain the following lemma.

LEMMA 2. *The following equality holds*

$$\mathrm{P}^k_{k\{\Delta\}}(\mathbf{y}, z) = \mathrm{P}^k_{k[\overline{\Delta}]}(\mathbf{y}, z).$$

Remark 2. Since the element $x_{i+1} \in R_i$ is a nonzerodivisor on (x_{i+1}) we get that $\mathrm{P}_{R_i}^{(x_{i+1}^2)} = y_{i+1}\mathrm{P}_{R_i}^{(x_{i+1})}$.

LEMMA 3. *The canonical projection $R_i \to R_{i+1}$ is a Golod homomorphism, and*

$$\frac{1}{\mathrm{P}^k_{R_{i+1}}} = \frac{1 - y_{i+1}z^2\mathrm{P}_{R_i}^{(x_{i+1})}}{\mathrm{P}^k_{R_i}}.$$

Proof. For the first statement we will mimic the proof of [Lev76, Theorem 2.3], and use the condition mentioned in Remark 1. The second statement then follows from Remark 2. Let X be a minimal DGA-resolution of k over R, consider the natural epimorphism $\eta\colon X \to X \otimes_R R/(x_{i+1}^2)$, and pick $z \in \tilde{Z}(X \otimes_R R/(x_{i+1}^2))$. Choose z' such that $\eta(z') = z$. Then $d(z') \in x_{i+1}^2 X$, and we write $d(z') = x_{i+1}^2 u$. Apply d once more, and we get

$$0 = d^2(z') = x_{i+1}d(x_{i+1}u).$$

By Remark 2 we get that $d(x_{i+1}u) = 0$. Let $T_{i+1} \in X_1$ such that $d(T_{i+1}) = x_{i+1}$, then $d(z' - T_{i+1}x_{i+1}u) = 0$, so since X is acyclic, we have a v such that $z' = T_{i+1}u + d(v)$. Thus $z = \eta(z') = \eta(T_{i+1})\eta(x_{i+1}u) + d\eta(v)$. Choose V by $V = \eta(T_{i+1})(X \otimes_R R/(x_{i+1}^2))$; since $T_{i+1}^2 = 0$, V satisfies the required condition, and the map $R_i \to R_{i+1}$ is Golod. ∎

LEMMA 4. *We have the identity*

$$\mathrm{P}^k_{k[\overline{\Delta}]}(y_1, \ldots, y_n, z) = \mathrm{P}^k_{k[\Delta]}\left(\frac{y_1}{1 - y_1 z}, \cdots, \frac{y_n}{1 - y_n z}, z\right).$$

Proof. Set $A = R_i/(x_{i+1})$, so that $\mathrm{P}^k_A = 1 + z\mathrm{P}_{R_i}^{(x_{i+1})}$. Since there are ring homomorphisms $A \to R_i \to A$ whose composition is the identity, we have $\mathrm{P}^k_{R_i} = \mathrm{P}^A_{R_i}\mathrm{P}^k_A$ by Herzog [Her77]. Using Lemma 3 we then get

$$\frac{1}{\mathrm{P}^k_{R_{i+1}}} = \frac{1 - y_{i+1}z^2\mathrm{P}^{(x_{i+1})}_{R_i}}{\mathrm{P}^k_{R_i}} = \frac{1 + y_{i+1}z - y_{i+1}z\mathrm{P}^A_{R_i}}{\mathrm{P}^k_{R_i}} = \frac{1 + y_{i+1}z}{\mathrm{P}^k_{R_i}} - \frac{y_{i+1}z}{\mathrm{P}^k_A}.$$

Note that $R_{i+1} \simeq A[x_{i+1}]/x_{i+1}I_\Gamma$, where Γ is the simplicial complex on the vertex set $[n] \setminus \{i+1\}$ defined by $\sigma \in \Gamma$ if and only if $\sigma \cup \{i+1\} \in \Delta$. From the proof of Backelin [Bac82, Lemma 1] we know that

$$\frac{1}{\mathrm{P}^k_{R_i}} = \frac{1}{\mathrm{P}^k_A} - \frac{z\left(1 + z\mathrm{P}^{I_\Gamma}_A\right)}{\mathrm{P}^k_A} \cdot \frac{y_{i+1}}{1 + y_{i+1}z}. \tag{2}$$

Using this formula to eliminate $\mathrm{P}^k_{R_i}$ from the expression for $1/\mathrm{P}^k_{R_{i+1}}$, we get

$$\frac{1}{\mathrm{P}^k_{R_{i+1}}} = \frac{1}{\mathrm{P}^k_A} - \frac{z\left(1 + z\mathrm{P}^{I_\Gamma}_A\right)}{\mathrm{P}^k_A} \cdot y_{i+1}. \tag{3}$$

Notice that the power series P^k_A and $\mathrm{P}^{I_\Gamma}_A$ do not involve the variable y_{i+1}. Thus, replacing $y_{i+1}/(1 - y_{i+1}z)$ transforms the right-hand side of (2) into that of (3). This establishes the equality

$$\mathrm{P}^k_{R_{i+1}}(y_1, \ldots, y_n, z) = \mathrm{P}^k_{R_i}\left(y_1, \ldots, y_i, \frac{y_{i+1}}{1 - y_{i+1}z}, y_{i+2}, \ldots, y_n, z\right).$$

Applying it successively with $i = 0, \ldots, n-1$ we get the assertion of the lemma. ∎

Proof of the theorem. Assume that $k[\Delta]$ is Golod; this is by definition equivalent to the equality

$$\mathrm{P}^k_{k[\Delta]}(\mathbf{y}, z) = \frac{\mathrm{P}^k_S(\mathbf{y}, z)}{1 - z\left(\mathrm{P}^{k[\Delta]}_S(\mathbf{y}, z) - 1\right)}.$$

According to Lemma 1, Lemma 2, and Lemma 4, this implies that

$$\begin{aligned}
\mathrm{P}^k_{k\{\Delta\}}(\mathbf{y}, z) &= \mathrm{P}^k_{k[\Delta]}\left(\frac{y_1}{1 - y_1z}, \cdots, \frac{y_n}{1 - y_nz}, z\right) \\
&= \frac{\mathrm{P}^k_S\left(y_1/(1 - y_1z), \ldots, y_n/(1 - y_nz), z\right)}{1 - z\left(\mathrm{P}^{k[\Delta]}_S\left(y_1/(1 - y_1z), \ldots, y_n/(1 - y_nz), z\right) - 1\right)} \\
&= \frac{\mathrm{P}^k_E(\mathbf{y}, z)}{1 - z\left(\mathrm{P}_E E^{k\{\Delta\}}(\mathbf{y}, z) - 1\right)},
\end{aligned}$$

so $k\{\Delta\}$ is Golod as well. By using that $y \mapsto y/(1 - yz)$ is invertible, with inverse $y \mapsto y/(1 + yz)$, we see that the converse is also true. ∎

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